Twist 3 of the $sl(2)$ sector of $N = 4$ SYM and reciprocity respecting evolution

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Abstract

We consider the bosonic $sl(2)$ sector of the maximally supersymmetric $\mathcal{N}=4$ SYM model and show that anomalous dimension of the twist-3 single-trace composite operators built of scalar fields, recently calculated up to the four-loop order, can be generated by a compact reciprocity respecting evolution kernel.

1 Introduction

QCD shares the vector boson — gluon — sector with supersymmetric Yang–Mills models (SYM). This suggests to explore supersymmetric partners of QCD in order to shed light on the subtle structure of the perturbative quark–gluon dynamics near the light cone. The latter manifests itself in parton distributions (structure functions of deep inelastic lepton–hadron scattering, DIS) and parton→hadron fragmentation functions (internal structure of jets in, e.g., e^+e^- annihilation into hadrons) which evolve with the hardness scale of the process, Q^2.

QCD is not an integrable quantum field theory. In spite of this, in certain sectors of the chromodynamics the integrability does emerge [1]. This happens, markedly, in the problem of high energy Regge behaviour of scattering amplitudes in the large N_c approximation (planar 't Hooft limit), in the spin $\frac{3}{2}$ baryon wave function, for the scale dependence of specific (maximal helicity) quasi-partonic operators (for review see [2]). What all these problems have in common, is the irrelevance of quark degrees of freedom and the dominance of the *classical* part of gluon dynamics, in the sense of the Low–Burnett–Kroll theorem [3].

A SYM dynamics exhibits the deeper integrability, the higher the symmetry. The maximally supersymmetric $\mathcal{N}=4$ YM theory occupies an exceptional position in this

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respect. A string of recent impressive theoretical developments [2, 4, 5] hints at an intriguing possibility that this QFT, super-conformally invariant at the quantum level, may admit an all loop solution for anomalous dimensions of its composite operators. QCD would benefit a lot from such a solution, since this would provide a one-line-all-loops representation for the *dominant* part of the perturbative gluon dynamics, the part that could be dubbed *LBK-classical*.

For the non-compact $\mathfrak{sl}(2)$ sector containing twist operators with arbitrarily high spin the three-loop Bethe Ansatz equations (BAE) have been conjectured in [6]. The perturbative expansion of the Staudacher BAE perfectly matches the *three-loop* anomalous dimensions of twist-2 Wilson operators that determine the Bjorken scaling violation pattern. At the same time, it apparently fails at the *fourth loop* where the wrapping problem is manifest [7].

Since in $\mathcal{N} = 4$ SYM all twist-2 operators belong to one super-multiplet, diagonalization of the matrices $\gamma(3 \times 3)$ and $\tilde{\gamma}(2 \times 2)$ describing evolution of unpolarized (g, λ, ϕ) and polarized parton distributions (g, λ) results in the relation [8]

$$\gamma_+(N+2) = \tilde{\gamma}_+(N+1) = \gamma_0(N) = \tilde{\gamma}_-(N-1) = \gamma_-(N-2) \;\equiv\; \gamma_{\rm uni}(N). \tag{1.1}$$

It expresses all five diagonal elements in terms of the unique function — the "universal anomalous dimension" $\gamma_{\rm uni}$ with shifted arguments. Inspired by the structure of the answer in the first two loops [9–12], Kotikov, Lipatov, Onishchenko and Velizhanin (KLOV) have proposed in [13] the *maximum transcendentality principle* according to which $\gamma^{(n)}_{\rm uni}$ at n loops is a linear combination *Euler–Zagier harmonic sums* of transcedentality $\tau = 2n - 1$. This allowed them to predict $\gamma^{(3)}_{\rm uni}$ by simply picking up from the three-loop non-singlet QCD anomalous dimension [14] the "most transcendental" terms ($\tau = 5$), having set $C_F = C_A = N_c$.

The complexity of higher loop expressions for $\gamma^{(n)}_{\rm uni}$, $n \geq 2$, is significantly reduced if one trades the anomalous dimension for a new object — the reciprocity respecting (RR) evolution kernel, [15]. In the case of $\mathcal{N}=4$ SYM ($\beta(\alpha) \equiv 0$) this trade-off takes an exceptionally simple form,

$$\gamma_\sigma(N) = \mathcal{P}(N - \tfrac{1}{2}\sigma\gamma_\sigma(N)), \quad \sigma = \begin{cases} +1, & \mathrm{T} \\ -1, & \mathrm{S} \end{cases}, \tag{1.2}$$

where $\gamma_\pm(N)$ are the anomalous dimensions responsible for time- and space-like parton evolution, respectively (in the S case which admits the operator product expansion, γ_- describes the scale dependence of Wilson operators of the leading twist 2). Here we employ the definition of the anomalous dimension which is more familiar to the integrable community[1]; within this convention the scaling violation rate is given by the expression

$$\frac{d}{d\ln Q^2} \ln D_N(Q^2) = -2\,\gamma(N)\,, \tag{1.3}$$

[1]The r.h.s. of (1.3) is traditionally called γ in the QCD parton evolution context.

with $D_N(Q^2)$ the Mellin moment of the parton distribution measured at the large momentum transfer (hardness) scale Q^2.

In the context of the QCD parton picture, the notion of the RR evolution kernel $\mathcal{P}$ emerges as a result of the reformulation of space-like (DIS, "S") and time-like parton multiplication processes (e^+e^-, "T") in terms of a unified evolution equation [16, 17]. This equation is constructed on a basis of the *parton fluctuation time* ordering. In the space of Bjorken (S)/Feynman (T) variable x conjugate to the Lorentz spin N,

$$\mathcal{P}(N) = \int_0^1 \frac{dx}{x}\, x^N \mathcal{P}(x) \equiv \mathcal{M}\left[\tilde{\mathcal{P}}(x)\right], \tag{1.4}$$

its integral kernel $\tilde{\mathcal{P}}(x)$, identical for the two channels, satisfies the Gribov–Lipatov reciprocity (GLR) [18] in all orders:

$$F(x) \;=\; -x\, F(x^{-1}). \tag{1.5}$$

Existence of the common evolution kernel $\mathcal{P}$ and its internal symmetry is reflected in the structure of the anomalous dimensions γ_σ that one finds solving (1.2).

Basso and Korchemsky in [19] approached this problem from the point of view of the large N expansion. They have generalised (1.2) to anomalous dimensions of quasi-partonic operators of arbitrary twist L, and traced its origin to the conformal symmetry.

Having analysed all anomalous dimensions of twist-2 operators, known to the two- and/or three-loop order in QCD and SYM theories, as well as in the scalar $\lambda\phi^4$ QFT (known at four loops), the authors demonstrated that the asymptotic series for the corresponding kernels $\mathcal{P}(N)$ run in integer negative powers of $N(N+1)$ — the quadratic Casimir of the collinear $SL(2;\mathbb{R})$ group. As a result, even, N^{-2n}, and odd terms, N^{-2n-1}, of the large N expansion of *anomalous dimensions* γ_σ turn out to be related. Basso and Korchemsky named this feature "parity preserving asymptotic series".

In $\mathcal{N}=4$ SYM the property of parity preserving series directly follows from the representation of the twist-2 kernel developed in three loops in [15]:

$$\mathcal{P}(N) = 4\, S_1(N) \cdot \left(\frac{N_c \alpha_{\rm ph}}{2\pi} + \widehat{\mathcal{A}}(N)\right) + \mathcal{B}(N). \tag{1.6}$$

Here $\alpha_{\rm ph}$ is the physical coupling which determines the strength of LBK-classical radiation [16, 20] and coincides with the so-called cusp anomalous dimension [21] whose all-order weak-coupling expansion in $\mathcal{N}=4$ SYM has been remarkably guessed in [22].

The functions $\widehat{\mathcal{A}} = \mathcal{O}\left(\alpha^3\right)$ and $\hat{\mathcal{B}} = \mathcal{O}\left(\alpha^2\right)$ have compact expressions in terms of RR combinations of complementary nested harmonic sums. Their asymptotic expansion at $N \to +\infty$ is *regular*, that is, contains no $\ln^p N$ enhancement factors at any level of the $1/N$ suppression. This makes the harmonic function S_1 the only source of the $\log N$ behaviour in (1.6), at least at the three-loop level.

This feature of the evolution kernel (1.6) is in a marked contrast with the anomalous dimension *per se*, whose large N expansion includes growing powers of $\log N$:

$$\gamma(N) = a \ln N + \sum_{k=0}^{\infty} \frac{1}{N^k} \sum_{m=0}^{k} a_{k,m} \ln^m N. \tag{1.7}$$

Physically, the reduction of singularity of the large N expansion is due to the fact that towers of subleading logarithmic singularities in the anomalous dimension are actually *inherited* from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RR evolution equation (1.2) generates them automatically[2] [15, 17].

In this letter we consider twist-3 operators in the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM and construct the evolution kernel $\mathcal{P}$ that generates the minimal anomalous dimension of single trace operators built of three scalar fields. This evolution kernel $\mathcal{P}$ satisfies the Gribov–Lipatov reciprocity, with x^2 substituted for x in eq. (1.5), in accord with the fact that the harmonic functions entering the twist-3 anomalous dimension have $N/2$ for the argument [7, 23]. We also demonstrate that the twist-3 kernel $\mathcal{P}$ admits the representation (1.6), with $\widehat{\mathcal{A}} = \mathcal{O}\left(\alpha^4\right)$.

This shows that the reciprocity respecting evolution, and thus the property of parity preserving series, manifest themselves also beyond the leading twist.

2 Reciprocity Respecting evolution kernel

2.1 Anomalous dimensions of twist-3 operators

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N} = 4$ SYM contains single trace states which are linear combinations of the basic operators

$$\mathrm{Tr}\left\{(\mathcal{D}^{s_1} Z) \cdots (\mathcal{D}^{s_L} Z)\right\}, \quad s_1 + \cdots + s_L = N, \tag{2.8}$$

where Z is one of the three complex scalar fields and $\mathcal{D}$ is a light-cone covariant derivative. The numbers $\{s_i\}$ are non-negative integers and N is the total spin. The number L of Z fields is the twist of the operator, *i.e.* the classical dimension minus spin. The subsector of states with fixed spin and twist is perturbatively closed under renormalization mixing.

The anomalous dimensions of states (2.8) are the eigenvalues $\gamma_L(N; g)$ of the dilatation operator — integrable Hamiltonian. These values were obtained by solving numerically the Bethe Ansatz equations (BAE), order by order in g^2, and guessing the answer in terms of harmonic sums of transcedentality $\tau = 2n - 1$, with n the number of loops. Since wrapping problems, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators [4, 24], the Bethe Ansatz equations for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

[2] for the space-like case under consideration, $\sigma = -1$

One finds [7,23]

$$\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\,, && (2.9a)\\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1 S_2) && (2.9b)\\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + S_1(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}), && (2.9c)\\
\gamma_3^{(4)} &= \tfrac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1}\\
&\quad -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3}\\
&\quad +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2}\\
&\quad +36\,S_{4,2,1} + 24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1} && (2.9d)\\
&\quad -24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} - 24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1}\\
&\quad -24\,S_{2,1,2,2} + 16\,S_{2,1,3,1} - 24\,S_{2,2,1,2} - 24\,S_{2,2,2,1} - 24\,S_{2,3,1,1}\\
&\quad -24\,S_{3,1,1,2} - 24\,S_{3,1,2,1} - 24\,S_{3,2,1,1} - 24\,S_{4,1,1,1} - 64\,S_{1,1,1,3,1}\\
&\quad -8\,\beta\,S_1\,S_3.
\end{aligned}$$

The last term in (2.9d), with $\beta = \zeta_3$, is the contribution from the dressing factor that appears in the BAE at the fourth loop [22].

The twist-3 anomalous dimension has two characteristic features:

1. All harmonic functions $S_{\mathbf{a}}$ are evaluated at half the spin, $S_{\mathbf{a}} \equiv S_{\mathbf{a}}\,(N/2)$. On the integrability side, this does not look unwarranted, since only *even* N belong to the non-degenerate ground state of the magnet.
2. No negative indices appear in (2.9), while in the case of twist-2 negative index sums were present starting from the second loop [13].

At the $N \to \infty$ limit, the *minimal* anomalous dimension γ (corresponding to the ground state) must exhibit the universal (LBK-classical) $\ln N$ behaviour which depends neither on the twist, nor on the nature of fields under consideration [25]. Computing analytically the large N expansion (1.7) yields (for $\beta = \zeta_3$)

$$\begin{aligned}
\gamma(N) &\simeq 4\,\mathrm{g}_{\mathrm{ph}}^2 \ln N,\\
\mathrm{g}_{\mathrm{ph}}^2 &\equiv \frac{N_c\,\alpha_{\mathrm{ph}}}{2\pi} = g^2 - \zeta_2\,g^4 + \tfrac{11}{5}\zeta_2^2\,g^6 - \left(\tfrac{73}{10}\zeta_2^3 + \zeta_3^2\right)g^8 + \ldots, && (2.10)
\end{aligned}$$

which matches the four-loop cusp anomalous dimension [22,26]. This is a non-trivial check, since the derivation of the expressions (2.9) was based on experimenting with finite values of the spin N.

Expanding (1.2) gives

$$\gamma = \mathcal{P} + \tfrac{1}{2}\mathcal{P}\dot{\mathcal{P}} + \tfrac{1}{4}\left[\mathcal{P}\dot{\mathcal{P}}^2 + \tfrac{1}{2}\mathcal{P}^2\ddot{\mathcal{P}}\right] + \tfrac{1}{8}\left[\mathcal{P}\dot{\mathcal{P}}^3 + \tfrac{3}{2}\mathcal{P}^2\dot{\mathcal{P}}\ddot{\mathcal{P}} + \tfrac{1}{6}\mathcal{P}^3\dddot{\mathcal{P}}\right] + \mathcal{O}\left(g^{10}\right) \qquad (2.11a)$$

and, equivalently,

$$\mathcal{P} = \gamma - \tfrac{1}{2}\gamma\dot{\gamma} + \tfrac{1}{4}\left[\gamma\dot{\gamma}^2 + \tfrac{1}{2}\gamma^2\ddot{\gamma}\right] - \tfrac{1}{8}\left[\gamma\dot{\gamma}^3 + \tfrac{3}{2}\gamma^2\dot{\gamma}\ddot{\gamma} + \tfrac{1}{6}\gamma^3\dddot{\gamma}\right] + \mathcal{O}\left(g^{10}\right), \qquad (2.11b)$$

where each dot marks derivative over N.

Substituting γ and $\mathcal{P}$ in terms of perturbative series in $g^2 = \frac{N_c \alpha}{2\pi}$,

$$\gamma = \sum_{n=1} g^{2\,n}\,\gamma^{(n)}, \qquad \mathcal{P} = \sum_{n=1} g^{2\,n}\,P^{(n)},$$

after a short calculation one obtains

$$P^{(1)} = 4\,\mathcal{S}_1, \tag{2.12a}$$

$$P^{(2)} = -2\,\mathcal{S}_3 - 4\,\zeta_2\,\mathcal{S}_1, \tag{2.12b}$$

$$\begin{aligned} P^{(3)} &= \mathcal{S}_5 + 2\,\zeta_2\,\mathcal{S}_3 + 4\,\left(\mathcal{S}_{3,2} + \mathcal{S}_{4,1} - 2\,\mathcal{S}_{3,1,1}\right) \\ &\quad + 4\,\mathcal{S}_1\left(2\,\mathcal{S}_{3,1} - \mathcal{S}_4 + 4\,\zeta_4\right) - 4\,\mathcal{S}_1^2\left(\mathcal{S}_3 - \zeta_3\right). \end{aligned} \tag{2.12c}$$

The fourth loop kernel we split into two terms, the first built up of harmonic functions $\mathcal{S}$ only, and the second one containing the zeta-function factors,

$$P^{(4)} = P_{\mathcal{S}}^{(4)} + P_{\zeta}^{(4)}. \tag{2.12d}$$

Here we present the corresponding expressions in terms of nested harmonic sums with *unity* indices moved to the head of the index vector, which form turns out to be more compact and is better suited for further transformation:

$$\begin{aligned} P_{\mathcal{S}}^{(4)} &= 8\left[-(\mathcal{S}_{3,3} + \mathcal{S}_{1,5} + 2\mathcal{S}_{2,4}) + 4(\mathcal{S}_{2,1,3} + \mathcal{S}_{1,2,3} + \mathcal{S}_{1,1,4}) - 8\mathcal{S}_{1,1,1,3}\right]\mathcal{S}_1 \\ &+ \tfrac{3}{2}\,\mathcal{S}_7 - 16\left(\mathcal{S}_{1,6} + \mathcal{S}_{4,3}\right) - 24\left(\mathcal{S}_{2,5} + \mathcal{S}_{3,4}\right) \\ &\quad + 48\left(\mathcal{S}_{1,1,5} + \mathcal{S}_{1,3,3} + \mathcal{S}_{3,1,3}\right) + 64\left(\mathcal{S}_{2,2,3} + \mathcal{S}_{2,1,4} + \mathcal{S}_{1,2,4}\right) \\ &\quad - 128\left(\mathcal{S}_{1,1,1,4} + \mathcal{S}_{2,1,1,3} + \mathcal{S}_{1,2,1,3} + \mathcal{S}_{1,1,2,3}\right) + 256\,\mathcal{S}_{1,1,1,1,3}; \end{aligned} \tag{2.12e}$$

$$\begin{aligned} P_{\zeta}^{(4)} &= 8\zeta_4\,\mathcal{S}_1^3 - 4\left[\zeta_2\zeta_3 + 8\zeta_5\right]\mathcal{S}_1^2 - \left[4(\zeta_3 + 2\beta)\mathcal{S}_3 + 49\zeta_6\right]\mathcal{S}_1 \\ &\quad + \left(8\mathcal{S}_{1,1,3} - 4\mathcal{S}_{1,4} - 4\mathcal{S}_{2,3} - \mathcal{S}_5\right)\zeta_2 - 8\mathcal{S}_3\,\zeta_4. \end{aligned} \tag{2.12f}$$

2.2 RR harmonic functions

In order to show that the twist-3 evolution kernel (2.12) satisfies the Gribov–Lipatov reciprocity, we need to introduce the corresponding basis of harmonic functions.

The functions

$$\frac{x}{x-1}\ln^{2k} x = \Gamma(2k+1)\cdot\tilde{\mathcal{S}}_{2k+1}(x)$$

satisfy the GL reciprocity (1.5) and generate harmonic functions with an odd index,

$$\frac{1}{\Gamma(2k+1)}\,\mathcal{M}\left[x\left(\frac{\ln^{2k} x}{x-1}\right)_+\right] = \mathcal{S}_{2k+1}(N); \tag{2.13a}$$

$$\frac{1}{\Gamma(2k+1)}\,\mathcal{M}\left[\frac{x}{x-1}\ln^{2k} x\right] = \widehat{\mathcal{S}}_{2k+1}(N) \equiv \mathcal{S}_{2k+1}(N) - \mathcal{S}_{2k+1}(\infty). \tag{2.13b}$$

In (2.13b) and hereafter the hat marks the function in the Mellin moment space with its value at $N=\infty$ subtracted: $\hat{a}(N) \equiv a(N) - a(\infty)$.

The inverse Mellin transforms $\tilde{\mathcal{S}}_{\mathbf{m}}(x)$ of multi-index sums $\mathcal{S}_{\mathbf{m}}(N)$ that enter eqs. (2.12c)–(2.12f) mix under the reciprocity operation (1.5), so we need to construct linear combinations of harmonic functions $\mathcal{S}$ that do have definite GL parity.

2.2.1 Integral representation

An index vector $\mathbf{m} = \{m_1, m_2, \ldots, m_\ell\}$ corresponds to transcedentality $\tau = \sum_{i=1}^{\ell} |m_i|$. We will refer to the transcedentality minus the length of the sum, $w[\mathbf{m}] = \tau - \ell$, as the *weight* of the harmonic function.

Consider the recurrence relation

$$\tilde{\Phi}_{b,\mathbf{m}}(x) = -[\Gamma(b)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz\,(z+1)}{z^2} \ln^{b-1}\frac{z}{x} \cdot \tilde{\Phi}_{\mathbf{m}}(z), \tag{2.14a}$$

where the single index function coincides with the image of the standard harmonic sum,

$$\tilde{\Phi}_a(x) = [\Gamma(a)]^{-1} \frac{x}{x-1} \ln^{a-1}\frac{1}{x} \;=\; \tilde{\mathcal{S}}_a(x). \tag{2.14b}$$

At the base of the recursion, (2.14b), we have

$$\tilde{\Phi}_a(x) \;=\; \Big(-x\,\tilde{\Phi}_a(x^{-1})\Big)\cdot(-1)^{a-1} \equiv \Big(-x\,\tilde{\Phi}_a(x^{-1})\Big)\cdot(-1)^{w[a]}.$$

An iteration (2.14a) increases the transcedentality of the function by b, and the length of the index vector by one, so that

$$w[\mathbf{m}] + b - 1 = w[b,\mathbf{m}].$$

Observing that the integration measure in (2.14a) transforms under $z \to 1/z$ as

$$d\phi(z) = \frac{dz\,(1+z)}{z^2}, \qquad d\phi(z^{-1}) = -z\,d\phi(z),$$

we conclude, by induction, that the functions $\tilde{\Phi}_{\mathbf{m}}(x)$ that the equation (2.14) generates, have definite GL parity determined by their weight:

$$\tilde{\Phi}_{\mathbf{m}}(x) \;=\; (-1)^{w[\mathbf{m}]}\cdot\Big(-x\,\tilde{\Phi}_{\mathbf{m}}(x^{-1})\Big). \tag{2.15}$$

Let us represent the r.h.s. of (2.14a) as

$$\frac{1}{\Gamma(b)}\frac{x}{x-1}\int_x^1 \frac{dz}{z}\ln^{b-1}\frac{z}{x}\cdot\frac{z-1}{z}\,\tilde{\Phi}_{\mathbf{m}}(z) \;-\; \frac{2}{\Gamma(b)}\frac{x}{x-1}\int_x^1 \frac{dz}{z}\ln^{b-1}\frac{z}{x}\,\tilde{\Phi}_{\mathbf{m}}(z). \tag{2.16}$$

Acting on a complementary harmonic sum[3] $\underline{\tilde{S}_{a,\mathbf{n}}}$ the first integral produces $\underline{\tilde{S}_{a+b,\mathbf{n}}}$ while the second generates (-2 times) the sum $\underline{\tilde{S}_{a,b,\mathbf{n}}}$, see (A.7). Iterating (2.14),

$$\tilde{\Phi}_{c,d} = \underline{\tilde{S}_{c+d}} - 2\,\underline{\tilde{S}_{c,d}}, \tag{2.17a}$$

$$\tilde{\Phi}_{b,c,d} = \underline{\tilde{S}_{b+c+d}} - 2\,(\underline{\tilde{S}_{b+c,d}} + \underline{\tilde{S}_{b,c+d}}) + 4\,\underline{\tilde{S}_{b,c,d}}, \tag{2.17b}$$

$$\begin{aligned}\tilde{\Phi}_{a,b,c,d} &= \underline{\tilde{S}_{a+b+c+d}} - 2\,(\underline{\tilde{S}_{a+b+c,d}} + \underline{\tilde{S}_{a+b,c+d}} + \underline{\tilde{S}_{a,b+c+d}}) \\ &+ 4\,(\underline{\tilde{S}_{a+b,c,d}} + \underline{\tilde{S}_{a,b+c,d}} + \underline{\tilde{S}_{a,b,c+d}}) - 8\,\underline{\tilde{S}_{a,b,c,d}}\,, \qquad \text{etc.}\end{aligned} \tag{2.17c}$$

In the Mellin moment space, the formal construction of functions $\Phi_{\mathbf{m}}(N)$ in terms of complementary harmonic sums is described in Appendix, see (A.11), (A.12).

Equation (2.15) shows that the functions $\tilde{\Phi}_{\mathbf{m}}$ with *even weight* $w[\mathbf{m}] = \tau - \ell$ are Reciprocity Respecting (RR).

2.3 Answer

In terms of the functions Φ so introduced the evolution kernel (2.12) reads

$$P^{(1)} = 4\,S_1, \tag{2.18a}$$

$$P^{(2)} = -4\,\zeta_2\,S_1 - 2S_3, \tag{2.18b}$$

$$P^{(3)} = \tfrac{44}{5}\,\zeta_2^2\,S_1 + 2\zeta_2\,S_3 + 3\,S_5 - 2\,\Phi_{1,1,3}, \tag{2.18c}$$

$$P^{(4)} = \left(\tfrac{146}{5}\zeta_2^3 + 4\zeta_3^2\right)S_1 - \zeta_2\left(3\,S_5 - 2\,\Phi_{1,1,3}\right) - \tfrac{24}{5}\,\zeta_2^2\,S_3 + 4\,S_1\,\widehat{\mathcal{A}}_4 + \mathcal{B}_4, \tag{2.18d}$$

where

$$\widehat{\mathcal{A}}_4 = 2\,\widehat{\Phi}_{1,1,1,3} - (\widehat{\Phi}_{1,5} + \widehat{\Phi}_{3,3}) - (2\beta - \zeta_3)\,\widehat{S}_3, \tag{2.19a}$$

$$\mathcal{B}_4 = 16\,\Phi_{1,1,1,1,3} - 4\left(\Phi_{3,1,3} + \Phi_{1,3,3} + \Phi_{1,1,5}\right) - \tfrac{5}{2}\,S_7, \tag{2.19b}$$

We recall that the arguments of the functions on the r.h.s. of the equations is $N/2$, and $\beta = \zeta_3$ in (2.19a). Alternatively, in terms of the physical coupling (2.10), the perturbative series for the kernel, $\mathcal{P} = \sum_{n=1} \mathbf{g}_{\rm ph}^{2n}\,\mathcal{P}_{\rm ph}^{(n)}$, takes the compact form

$$\mathcal{P}_{\rm ph}^{(1)} = 4\,S_1, \tag{2.20a}$$

$$\mathcal{P}_{\rm ph}^{(2)} = -2\,S_3, \tag{2.20b}$$

$$\mathcal{P}_{\rm ph}^{(3)} = 3\,S_5 - 2\,\Phi_{1,1,3} \;+\; \zeta_2\cdot(-2\,S_3), \tag{2.20c}$$

$$\mathcal{P}_{\rm ph}^{(4)} = 4\,S_1\cdot\widehat{\mathcal{A}}_4 + \mathcal{B}_4 \;+\; 2\,\zeta_2\cdot\left(3\,S_5 - 2\,\Phi_{1,1,3}\right). \tag{2.20d}$$

Since all harmonic functions involved have *even* weights w, the splitting kernel (2.20) is Reciprocity Respecting. Mark that given the argument $N/2$ of the harmonic functions in (2.20), (2.19), the reciprocity relation (1.5) applies to $\tilde{\mathcal{P}}(x^2)$.

This result can be compared with the evolution kernel[4] that generates the twist-2

[3]See Appendix for the definition and properties of complementary harmonic sums

[4]the GL parity of a harmonic function with k negative indices is $(-1)^{w+k}$

universal anomalous dimension [15]:

$$
\begin{aligned}
\mathcal{P}_{\rm ph}^{(1)} &= 4\,\mathcal{S}_1(N); && (2.21a)\\
\mathcal{P}_{\rm ph}^{(2)} &= -4\,\mathcal{S}_3(N) + 4\,\Phi_{1,-2}(N); && (2.21b)\\
\mathcal{P}_{\rm ph}^{(3)} &= 8\,\mathcal{S}_5(N) - 24\,\Phi_{1,1,1,-2}(N) - 8\,\zeta_2\,\mathcal{S}_3(N) \\
&- 8\,\mathcal{S}_1(N)\cdot\big[2\,\widehat{\Phi}_{1,1,-2}(N) + \widehat{\Phi}_{-2,-2}(N) - \widehat{\mathcal{S}}_{-4}(N) + \zeta_2\,\widehat{\mathcal{S}}_{-2}(N)\big]. && (2.21c)
\end{aligned}
$$

2.4 Asymptotic expansion

Large Mellin moments N correspond to parton light-cone momentum fractions $x \to 1$. In this region only partons with small energy–momentum can be produced in the final state of a hard process, $(1-x) \ll 1$, and radiation of soft gluons dominates the answer. Therefore, the large N behaviour is important for understanding how the LBK physics manifests itself in the structure of anomalous dimensions and of the evolution kernel.

At large positive values of the argument, $z \to +\infty$, the harmonic function $\mathcal{S}_1(z)$ has the asymptotic expansion

$$
S_1(z) = \psi(z+1) - \psi(1) = (\ln y + \gamma_E) + \tfrac{1}{6}y^{-2} - \tfrac{1}{30}y^{-4} + \tfrac{4}{315}y^{-6} + \ldots, \qquad (2.22a)
$$

where

$$
y^2 = z(z+1). \qquad (2.22b)
$$

Such a structure of the series is inherited by all harmonic sums $\mathcal{S}_a(z)$, which can be obtained from (2.22) by simple differentiation,

$$
\widehat{\mathcal{S}}_a(z) = \frac{(-1)^{a-1}}{\Gamma(a)}\left(\frac{d}{dz}\right)^{a-1} \mathcal{S}_1(z), \qquad \frac{d}{dz} = \frac{dy}{dz}\,\frac{d}{dy} = \sqrt{1+\frac{1}{4\,y^2}}\,\frac{d}{dy},
$$

as well as by the nested harmonic functions $\Phi_{\mathbf{m}}(z)$. Namely, the asymptotic series of the functions of even (odd) weight contain only even (odd) inverse powers of y. This is a direct consequence of their inverse Mellin images having definite GL-parity [19].

2.4.1 Conformal structure of the large N expansion

This feature of the asymptotic expansion has a clear symmetry origin [19].

On the light-cone the residual conformal invariance is that of the collinear subgroup of the conformal group $SL(2;\mathbb{R}) \subset SO(2,4)$ [27]. In the $\mathcal{N} = 4$ SYM theory, (super)conformal invariance is exact at all loops and quasipartonic operators can be classified according to the representations of $SL(2;\mathbb{R})$. Also, the operators belonging to different $SL(2;\mathbb{R})$ multiplets do not mix under renormalization. The anomalous

dimension of each conformal multiplet depends only on the conformal spin characterizing the representation

$$j(g) = \tfrac{1}{2}(N + \Delta(g)) = N + \tfrac{1}{2}\,L + \tfrac{1}{2}\,\gamma_L(N), \tag{2.23}$$

where $\Delta(g)$ is the scaling dimension, $\Delta = N + L + \gamma_L$, which includes the quantum anomalous contribution $\gamma_L(N; g)$. Hence, the spin dependence of the anomalous dimension at fixed twist L is encoded in the non-linear relation

$$\gamma_L(N) = f_L\left(N + \tfrac{1}{2}\,\gamma_L(N)\right), \tag{2.24}$$

which coincides with the relation (1.2) between the anomalous dimension and the Reciprocity Respecting evolution kernel $\mathcal{P}$ for the space-like evolution, $\sigma = -1$.

As we have mentioned in the Introduction, the fact that the asymptotic series for the RR evolution kernel runs in integer powers of y^2 (modulo logarithmic corrections) has been dubbed "parity preserving asymptotic series" and verified for a broad variety of QFT models in [19] for the twist-2 case. Now we see that this property holds for twist-3 as well.

In general, the conformal Casimir operator reads [28, 29]

$$J^2 = (N + L\,\eta - 1)(N + L\,\eta), \tag{2.25}$$

where $\eta = \frac{1}{2}, 1, \frac{3}{2}$ for scalars, gaugino, and gauge boson fields, respectively.

For twist-2, the argument of harmonic functions in eqs. (2.21) coincides with the Lorentz spin, $z = N$, and we have simply $J^2 = N(N+1) = y^2$. For the twist-3 in the scalar sector, (2.20), we have $z = N/2$; substituting $L = 3$, $\eta = \frac{1}{2}$ into (2.25) gives

$$J^2 = (N + \tfrac{1}{2})(N + \tfrac{3}{2}) = 4\,(\tfrac{N}{2} + \tfrac{1}{4})(\tfrac{N}{2} + \tfrac{3}{4}) = 4\,z^2 + 4\,z + \tfrac{3}{4} = 4\,y^2 + \tfrac{3}{4}.$$

Thus, the expansion in y^2 translates into that in the quadratic Casimir, J^2.

Asymptotic series for the evolution kernel, both for the twist-2 and twist-3 operators, can be cast in the form

$$\mathcal{P}(N) = (\ln y + \gamma_E)\left[4\,\mathrm{g}_{\mathrm{ph}}^2 + \sum_{n=1}^{\infty}\frac{a_n}{y^{2\,n}}\right] + \sum_{m=0}^{\infty}\frac{b_m}{y^{2\,m}}\,, \qquad \left\{\begin{array}{ll} y^2 = N(N+1), & L = 2; \\ y^2 = \frac{N}{2}(\frac{N}{2}+1), & L = 3. \end{array}\right. \tag{2.26}$$

where the L-dependent coefficients a, b are given in series of the coupling.

2.4.2 Logarithmic structure of the large N expansion

Apart from "parity preservation", the expansion (2.26) has another remarkable feature. In a sharp contrast with the series for the *anomalous dimension*, where the number of logs increases with the power of the $1/N$ suppression, see (1.7), the *evolution kernel* appears to be *linear* in $\ln N$.

Whether this property holds beyond the known orders (g^6 for $L=2$; g^8 for $L=3$) remains unknown. Should this be the case, that would mean that the tower of subleading logarithmic singularities in the anomalous dimension are actually *inherited*, in all orders, from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RR evolution equation (1.2) generates them automatically.

The leading logarithmic behaviour is universal and is given, in terms of the physical coupling (2.10), by the LBK-classical contribution to the evolution kernel,

$$\mathcal{P}(N) \simeq 4\,\mathrm{g}_{\mathrm{ph}}^2\, \mathcal{S}_1(z) = 4\,\mathrm{g}_{\mathrm{ph}}^2 \ln N + \mathcal{O}(1)\,. \tag{2.27}$$

The non-trivial N^2 dependence accompanying the $\log N$ enhancement, $\delta\mathcal{P} \propto \log N/N^2$, appears only at the level of $a_1 \propto g^6$ and $a_1 \propto g^8$ for twist-2 and 3, correspondingly.

It follows from the structure of the relation (2.11a),

$$\gamma(N) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\frac{d}{dN}\right)^{k-1} \left[\mathcal{P}(N)\right]^k, \tag{2.28}$$

that the maximally logarithmically enhanced contributions to the anomalous dimension, $\log^k N/N^k$, are driven by the leading, first loop term (2.27). Representing the twist-3 anomalous dimension as

$$\gamma(N; L\!=\!3) = a_0\left(\ln\frac{N}{2} + \gamma_E\right) + B + \sum_{k=1}^{\infty}\frac{1}{N^k}\sum_{m=0}^{k} a_{k,m}\ln^m N, \qquad a_0 = 4\,\mathrm{g}_{\mathrm{ph}}^2, \tag{2.29}$$

and substituting (2.27) into (2.28) we immediately obtain

$$a_{1,1} = \tfrac{1}{2}a_0^2\,, \quad a_{2,2} = -\tfrac{1}{8}a_0^3\,, \quad a_{3,3} = \tfrac{1}{24}a_0^4\,; \qquad a_{k,k} = \frac{(-1)^{k-1}}{2^k\,k}\,a_0^{k+1}. \tag{2.30}$$

In order to predict the next-to-maximal logarithmic terms, $a_{k,k-1}\log^{k-1}N/N^k$, in the expansion (2.29), it suffices to keep the constant in (2.27) and use the approximation

$$\begin{aligned}\mathcal{P}(N) &\simeq 4\,\mathrm{g}_{\mathrm{ph}}^2\,\mathcal{S}_1(z) + B = 4\,\mathrm{g}_{\mathrm{ph}}^2\cdot\left(\ln\sqrt{z(z+1)} + \gamma_E\right) + B + \mathcal{O}\left(z^{-2}\right)\\ &\simeq 4\,\mathrm{g}_{\mathrm{ph}}^2\left[\ln z + \gamma_E + \frac{1}{2z}\right] + B,\end{aligned} \tag{2.31a}$$

where

$$B = B(g^2, L\!=\!3) = -2\zeta_3\,\mathrm{g}_{\mathrm{ph}}^4 - (2\zeta_2\zeta_3 + \zeta_5)\,\mathrm{g}_{\mathrm{ph}}^6 + \mathcal{O}\left(\mathrm{g}_{\mathrm{ph}}^8\right). \tag{2.31b}$$

In particular, this gives (recall that for twist-3 we have $z = N/2$)

$$a_{1,0} = a_0\left(1 + \tfrac{1}{2}B\right). \tag{2.32}$$

Relations (2.30), (2.32) hold in all orders in the coupling constant.

3 Conclusions

The notion of the *reciprocity respecting* evolution equation (RREE) emerged in an attempt to combine, in a single framework, anomalous dimensions of space-like parton distributions and time-like parton fragmentation functions, in order to simplify the structure of the higher order corrections. The basic observation is that the complexity of higher loop contributions is, to a large extent, *inherited* from lower orders. This is especially so for the major part of the QCD anomalous dimensions which is governed by the "classical" gluon radiation, in the sense of the Low–Burnett–Kroll (LBK) theorem. One may argue that there should exist a framework in which effects of the classical gluon fields would be fully generated, in all orders, from the *first loop*, in the spirit of the LBK wisdom.

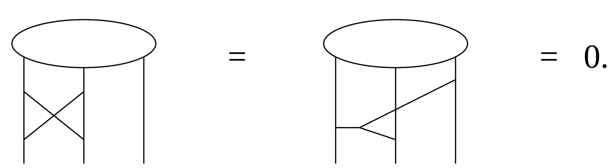
In the space of Mellin moments of parton distributions, the corresponding part of the anomalous dimension, in the nth order of the perturbative loop expansion, is described by harmonic functions of the *maximal transcedentality*, $\tau = 2n - 1$.

Since QCD shares the gauge boson sector with supersymmetric theories, the latter provide a perfect ground for testing the inheritance idea. The maximally supersymmetric $\mathcal{N}=4$ SYM model is exceptional in this respect. In this theory *quantum effects* due to gluons, gauginos and scalars cancel in the beta-function (in all orders) as well as in the one-loop anomalous dimension.

In this paper we have addressed the question of applicability of the notion of RREE beyond the leading twist. We considered the bosonic $sl(2)$ sector of the $\mathcal{N}=4$ SYM and found that the Gribov–Lipatov reciprocity relation holds in four loops for the evolution kernel $\mathcal{P}$ describing the minimal (ground state) anomalous dimension of twist-3 operators built of three scalar fields. The GL relation holds with x^2 substituted for x, in accord with the fact that the harmonic functions entering the twist-3 anomalous dimension have $N/2$ as the argument.

The twist-3 evolution kernel is given in eqs. (2.20), (2.19). Having derived this kernel from the space-like anomalous dimension, prediction for the *time-like* twist-3 anomalous dimension follows immediately from the RREE. It is given by (2.11a) with the r.h.s. changed to $-$r.h.s.$[-\mathcal{P}]$ (i.e., changing signs of the terms even in $\mathcal{P}$).

The structure of the twist-3 evolution kernel turns out to be even simpler than that for the leading twist-2, (2.21). In particular, an extreme simplicity of the second loop kernel (2.20b) is likely to be due to the fact that the genuine $\mathcal{O}\left(\alpha^2\right)$ graphs that renormalise twist-3 operators *vanish* identically because of the colour structure (each pair in the adjoint representation):



In the large N limit, where classical LBK gluon radiation dominates, the asymptotic series for evolution kernels are much less singular than the corresponding series for the anomalous dimensions. Namely, only the first power of $S_1 \propto \ln N$ is present in the evolution kernel, both for twist-3 (2.20), and twist-2 (2.21). Whether this remarkable property is of general nature and holds in higher orders ($n \geq 4$ for twist-2, and $n \geq 5$ for twist-3) remains unknown.

In twist-3, the subleading logarithmic enhancement $S_1 \cdot \widehat{\mathcal{A}}_4(N) \propto \ln N/N^2$ appears for the first time at the level of α^4, while in the case of twist-2 such term is present in the third loop, $\sim \alpha^3 \ \ln N/N^2$.

The specific features of the minimal twist-3 anomalous dimension, and of the corresponding evolution kernel, such as the absence of negative indices and appearance of $N/2$ as the argument of harmonic structures are awaiting physical explanation.

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A RR harmonic structures

A.1 Complementary harmonic sums

Let $\mathbf{m}$ denote a string of integers, $\mathbf{m} = \{m_1, m_2, \ldots, m_\ell\}$. We define multi-index (or "nested") complementary harmonic sums $\underline{S_m}$ by the recursive relation

$$\underline{S_{a,\mathbf{m}}}(N) = S_{a,\mathbf{m}}(N) - S_a(N) \cdot \underline{S_\mathbf{m}}(\infty), \tag{A.1a}$$

where we set

$$\underline{S_a}(N) \equiv S_a(N), \tag{A.1b}$$

For any index vector $\mathbf{m}$ with the last index $m_\ell \neq 1$, the value $\underline{S_\mathbf{m}}(\infty)$ is finite, so that we can carry out subtraction to introduce

$$\underline{\widehat{S}_\mathbf{m}}(N) \equiv \underline{S_\mathbf{m}}(N) - \underline{S_\mathbf{m}}(\infty). \tag{A.2}$$

For positive integer N we have

$$\underline{\widehat{S}_{a,\mathbf{m}}}(N) = -\sum_{n=N+1}^{\infty} (\operatorname{sgn} a)^n \, n^{-|a|} \, \underline{\widehat{S}_\mathbf{m}}(n). \tag{A.3}$$

From now on we restrict ourselves to positive indices[5] so that for the index vector $\mathbf{m}=\{m_1, m_2, \ldots, m_\ell\}$, with $m_\ell \geq 2$,

$$\underline{\mathcal{S}_{\mathbf{m}}}(N) \;=\; (-1)^{\ell-1} \sum_{n_1=1}^{N} n_1^{-m_1} \sum_{n_2=n_1+1}^{\infty} n_2^{-m_2} \ldots \sum_{n_\ell=n_{\ell-1}+1}^{\infty} n_\ell^{-m_\ell}, \tag{A.4a}$$

$$\underline{\widehat{\mathcal{S}}_{\mathbf{m}}}(N) \;=\; (-1)^{\ell} \sum_{n_1=N+1}^{\infty} n_1^{-m_1} \sum_{n_2=n_1+1}^{\infty} n_2^{-m_2} \ldots \sum_{n_\ell=n_{\ell-1}+1}^{\infty} n_\ell^{-m_\ell}. \tag{A.4b}$$

A.1.1 Large N behaviour

As follows from their definition (A.4), (subtracted) complementary sums fall in the $N \to \infty$ limit. The leading power of the large N behaviour is given by the *weight* of the sum, w (equal transcedentality minus length):

$$\underline{\widehat{\mathcal{S}}_{\mathbf{m}}}(N) \propto N^{-w[\mathbf{m}]}, \qquad w[\mathbf{m}] \equiv \tau - \ell \equiv \sum_{i=1}^{\ell}(m_i - 1). \tag{A.5}$$

Importantly, large N asymptotic expansion of complementary sums contains no $\log N$ enhanced terms, at any order of the $1/N$ suppression.

A.1.2 Mellin image

Inverse Mellin images of complementary sums,

$$\underline{\tilde{\mathcal{S}}_{\mathbf{m}}}(x) = \mathcal{M}^{-1}\left[\, \underline{\widehat{\mathcal{S}}_{\mathbf{m}}}(N) \,\right], \tag{A.6}$$

can be generated by the integral operation

$$\underline{\tilde{\mathcal{S}}_{a,\mathbf{m}}}(x) = [\Gamma(a)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz}{z} \ln^{a-1}\frac{z}{x} \cdot \underline{\tilde{\mathcal{S}}_{\mathbf{m}}}(z). \tag{A.7}$$

In particular,

$$\underline{\tilde{\mathcal{S}}_{b,a}}(x) \;=\; [\Gamma(a)\Gamma(b)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz}{z-1} \ln^{b-1}\frac{z}{x} \ln^{a-1}\frac{1}{z}, \tag{A.8a}$$

$$\underline{\tilde{\mathcal{S}}_{c,b,a}}(x) \;=\; [\Gamma(a)\Gamma(b)\Gamma(c)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dy}{y-1} \ln^{c-1}\frac{y}{x} \int_y^1 \frac{dz}{z-1} \ln^{b-1}\frac{z}{y} \ln^{a-1}\frac{1}{z}, \tag{A.8b}$$

etc.

[5]Generalisation is straightforward; for complementary sums with a negative index see [15].

A.1.3 Derivative over N

$$\frac{d}{dN}\,\underline{\mathcal{S}_{\mathbf{m}}}(N) = \mathcal{M}\left[\ln x \cdot \tilde{\underline{\mathcal{S}_{\mathbf{m}}}}(x)\right] = -\sum_{k=1}^{\ell} m_k \cdot \underline{\widehat{\mathcal{S}}_{m_1,\ldots,m_k,m_k+1,m_{k+1},\ldots,m_\ell}}(N), \qquad \text{(A.9)}$$

Using the definition (A.1a), it is straightforward to derive the formula for the derivative of the standard nested harmonic sums,

$$\frac{d}{dN}\,\mathcal{S}_{\mathbf{m}}(N) = -\sum_{k=1}^{\ell} m_k\,\widehat{\mathcal{S}}_{m_1,\ldots,m_k+1,\ldots,m_\ell}(N) + m_\ell \sum_{k=1}^{\ell} \widehat{\mathcal{S}}_{m_1,\ldots,m_k}(N)\cdot \underline{\mathcal{S}_{m_{k+1},\ldots,m_\ell}}(\infty). \qquad \text{(A.10)}$$

A.2 Construction

Given a complementary sum $\underline{\mathcal{S}_{\mathbf{m}}}$ of length ℓ, we construct a combination of sums $y^{(r)}$ having the same transcedentality and *reduced length*. The index vector $\mathbf{m}$ has ℓ indices, separated by $\ell-1$ commas; by erasing $(\ell-r)$ of them, and summing up the non-separated indices, we construct

$$\begin{aligned}
y_{\mathbf{m}}^{(\ell)} &= \underline{\mathcal{S}_{\mathbf{m}}} \equiv \underline{\mathcal{S}_{m_1,m_2,m_3,\ldots,m_\ell}}, && \text{(A.11a)}\\
y_{\mathbf{m}}^{(\ell-1)} &= \underline{\mathcal{S}_{[m_1+m_2],m_3,\ldots,m_\ell}} + \underline{\mathcal{S}_{m_1,[m_2+m_3],\ldots,m_\ell}} + \ldots + \underline{\mathcal{S}_{m_1,m_2,\ldots,[m_{\ell-1}+m_\ell]}}, && \text{(A.11b)}\\
y_{\mathbf{m}}^{(\ell-2)} &= \underline{\mathcal{S}_{[m_1+m_2+m_3],m_4,\ldots,m_\ell}} + \underline{\mathcal{S}_{[m_1+m_2],[m_3+m_4],\ldots,m_\ell}} \\
&\quad + \underline{\mathcal{S}_{[m_1+m_2],m_3,[m_4+m_5],\ldots,m_\ell}} + \ldots + \underline{\mathcal{S}_{m_1,m_2,\ldots,[m_{\ell-2}+m_{\ell-1}+m_\ell]}}, && \text{(A.11c)}\\
&\ldots \\
y_{\mathbf{m}}^{(2)} &= \sum_{k=1}^{\ell-1} \underline{\mathcal{S}_{[\sum_1^k m_i],[\sum_{k+1}^{\ell} m_j]}}, && \text{(A.11d)}\\
y_{\mathbf{m}}^{(1)} &= \underline{\mathcal{S}_{[\sum_1^\ell m_i]}} = \mathcal{S}_\tau. && \text{(A.11e)}
\end{aligned}$$

Now, we consider definite combinations of complementary harmonic sums:

$$\Phi_{\mathbf{m}} = \sum_{r=1}^{\ell} (-2)^{r-1} y_{\mathbf{m}}^{(r)} = (-2)^{\ell-1}\underline{\mathcal{S}_{\mathbf{m}}} + \ldots + 4y_{\mathbf{m}}^{(3)} - 2y_{\mathbf{m}}^{(2)} + \mathcal{S}_\tau. \qquad \text{(A.12)}$$

Specificity of the combinations so organised lies in the structure of their inverse Mellin images. Namely, inverse Mellin images of (subtracted) harmonic functions (A.12),

$$\tilde{\hat{\Phi}}_{\mathbf{m}}(x) = \mathcal{M}^{-1}\left[\hat{\Phi}_{\mathbf{m}}(N)\right], \qquad \text{(A.13)}$$

have definite GL parity,

$$\tilde{\hat{\Phi}}_{\mathbf{m}}(x) \;=\; (-1)^w \cdot \Big(-x\,\tilde{\hat{\Phi}}_{\mathbf{m}}(x^{-1})\Big). \qquad \text{(A.14)}$$

This shows that the functions $\tilde{\hat{\Phi}}_{\mathbf{m}}$ with *even weight* $w[\mathbf{m}] = \tau - \ell$ are RR.

A.2.1 Some useful examples

$\tau = \mathbf{5}$:

$$\Phi_{1,1,3} = 4\,\underline{\mathcal{S}_{1,1,3}} - 2(\underline{\mathcal{S}_{1,4}} + \underline{\mathcal{S}_{2,3}}) + \mathcal{S}_5. \tag{A.15}$$

$\tau = \mathbf{6}$:

$$\Phi_{3,3} = -2\,\underline{\mathcal{S}_{3,3}} + \mathcal{S}_6; \tag{A.16a}$$
$$\Phi_{1,5} = -2\,\underline{\mathcal{S}_{1,5}} + \mathcal{S}_6; \tag{A.16b}$$
$$\Phi_{1,1,1,3} = -8\,\underline{\mathcal{S}_{1,1,1,3}} + 4(\underline{\mathcal{S}_{1,1,4}} + \underline{\mathcal{S}_{1,2,3}} + \underline{\mathcal{S}_{2,1,3}}) - 2(\underline{\mathcal{S}_{1,5}} + \underline{\mathcal{S}_{2,4}} + \underline{\mathcal{S}_{3,3}}) + \mathcal{S}_6. \tag{A.16c}$$

$\tau = \mathbf{7}$:

$$\Phi_{1,3,3} = 4\,\underline{\mathcal{S}_{1,3,3}} - 2(\underline{\mathcal{S}_{1,6}} + \underline{\mathcal{S}_{4,3}}) + \mathcal{S}_7; \tag{A.17a}$$
$$\Phi_{3,1,3} = 4\,\underline{\mathcal{S}_{3,1,3}} - 2(\underline{\mathcal{S}_{4,3}} + \underline{\mathcal{S}_{3,4}}) + \mathcal{S}_7; \tag{A.17b}$$
$$\Phi_{1,1,5} = 4\,\underline{\mathcal{S}_{1,1,5}} - 2(\underline{\mathcal{S}_{1,6}} + \underline{\mathcal{S}_{2,5}}) + \mathcal{S}_7; \tag{A.17c}$$
$$\begin{aligned}\Phi_{1,1,1,1,3} = {} & 16\,\underline{\mathcal{S}_{1,1,1,1,3}} - 8(\underline{\mathcal{S}_{1,1,1,4}} + \underline{\mathcal{S}_{1,1,2,3}} + \underline{\mathcal{S}_{1,2,1,3}} + \underline{\mathcal{S}_{2,1,1,3}}) \\ & + 4(\underline{\mathcal{S}_{1,1,1,4}} + \underline{\mathcal{S}_{1,1,2,3}} + \underline{\mathcal{S}_{1,2,1,3}} + \underline{\mathcal{S}_{2,1,1,3}}) \\ & - 2(\underline{\mathcal{S}_{1,6}} + \underline{\mathcal{S}_{2,5}} + \underline{\mathcal{S}_{3,4}} + \underline{\mathcal{S}_{4,3}}) + \mathcal{S}_7.\end{aligned} \tag{A.17d}$$

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